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ON STABILIZATION OF THE ROTATIONAL MOTION OF A SOLID WITH FLYWHEELS IN A NEWTONIAN FORCE FIELD

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V. V. KREMENTULO
(Moscow)
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A solution of the problem of optimal stabilization (in a specific sense) of the rotational motion of a gyrostat (a solid with two flywheels) in a central Newtonian force field is given within the framework of analytical control theory [1].

1. Initial equations of motion. Formulation of the problem. Retaining the notation used earlier [2], let us consider a solid along two of whose principal axes of inertia are located the axes of rotation of homogeneous symmetric flywheels, set in motion by special motors. The gyrostat is in a central Newtonian force field (O_1 is the attracting center, and O is the center of mass of the gyrostat).

Shown in Fig. 1 are the following coordinate systems: $O_1X_1X_2X_3$ — the inertial system, $Ox_1x_2x_3$ — rigidly coupled to the gyrostat and directed along its principal axes of inertia (Ox_1 and Ox_2 are the axes of flywheel rotation), $Ox_1'x_2'x_3'$ — semi-mobile (the Ox_3' axis coincides with the Ox_3 axis, while the Ox_1', Ox_2' axes do not take part in gyrostat rotation around the Ox_3 axis). Let us introduce the notation: C_1, C_2, C_3 are the gyrostat moments of inertia relative to the $Ox_1x_2x_3$ axes, respectively, J_1, J_2 are the axial moments of flywheel inertia (for a symmetric gyrostat $C_1 = C_2 = C$, $J_1 = J_2 = J$); q_1, q_2, q_3 are the projections of the instantaneous angular velocity of the trihedral $Ox_1'x_2'x_3'$ on these axes, β_{ik} are the direction cosines of the angles between the $O_1X_1X_2X_3$ and $Ox_1'x_2'x_3'$ axes, h_1, h_2, h_3 are projections of the gyrostat kinetic moment vector relative to the center O_1 on the $O_1X_1X_2X_3$ axis, u_1, u_2 are the control moments around the flywheel axes Ox_1, Ox_2 produced by the motors, M is the gyrostat mass, X_1, X_2, X_3 are coordinates of the center of gyrostat mass in the $O_1X_1X_2X_3$ system, U is the gravitational force function of the form for a symmetric gyrostat [2, 3] ($\varkappa$ is the gravitational constant)

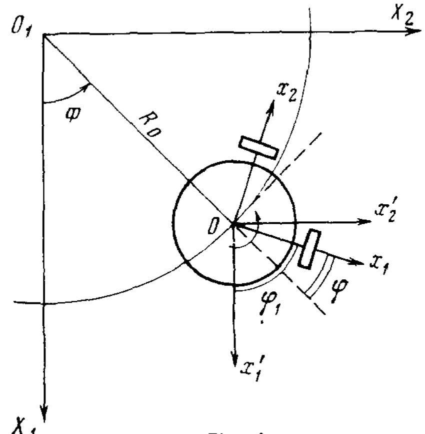

Fig. 1

$$U = \frac{\varkappa M}{R} + \frac{1}{2}\frac{\varkappa}{R^3}(C_3 - C) - \quad (1.1)$$

$$\frac{3}{2}\frac{\varkappa}{R^3}(C_3 - C)(X_1\beta_{13} + X_2\beta_{23} + X_3\beta_{33})^2$$

$$(R = \sqrt{X_1^2 + X_2^2 + X_3^2})$$

The stationary mode being studied is motion of regular precession type: the gyrostat center of mass O moves in the $X_1O_1X_2$ plane along a circular orbit of radius R_0 at the constant angular velocity $\Phi^\cdot = \omega_1$; the gyrostat moves uniformly with the relative angular velocity $\varphi^\cdot = \omega$ around the axis of symmetry Ox_3 directed perpendicularly to the plane of the orbit; the control motors are hence disconnected, and the flywheels do not rotate with respect to the body.

The equations of gyrostat motion can be represented as [2]

$$MX_i^{\cdot\cdot} = \frac{\partial U}{\partial X_i} \qquad (i = 1, 2, 3) \qquad (1.2)$$

$$(C - J)\, q_1^\cdot = -(C - J)\, q_2\varphi_1^\cdot + (q_3 + \varphi_1^\cdot) \sum (h_i - L_i)\, \beta_{i2} -$$
$$q_2 \sum (h_i - L_i)\, \beta_{i3} + M_{x_1'} - w_1$$

$$(C - J)\, q_2^\cdot = (C - J)\, q_1\varphi_1^\cdot - (q_3 + \varphi_1^\cdot) \sum (h_i - L_i)\, \beta_{i1} +$$
$$q_1 \sum (h_i - L_i)\, \beta_{i3} + M_{x_2'} - w_2$$

$$C_3 (q_3 + \varphi_1^\cdot)^\cdot = q_2 \Sigma (h_i - L_i)\, \beta_{i1} - q_1 \Sigma (h_i - L_i)\, \beta_{i2}$$

$$\beta_{i1}^\cdot + q_2\beta_{i3} - q_3\beta_{i2} = 0 \qquad (i = 1, 2, 3) \quad (1\ 2\ 3)$$

Here w_1, w_2 denote new control moments relative to the Ox_1', Ox_2' axes, L_i are the projections of the kinetic moment of the center of mass, and $M_{x_1'}$, $M_{x_2'}$ are the moments of the gravitational forces on the basis of (1.1)

$$w_1 = u_1 \cos \varphi_1 - u_2 \sin \varphi_1, \quad w_2 = u_1 \sin \varphi_1 + u_2 \cos \varphi_1 \qquad (1.3)$$
$$(\varphi_1^\cdot = \varphi^\cdot + \Phi^\cdot \beta_{33})$$
$$L_1 = M\,(X_2X_3^\cdot - X_2^\cdot X_3) \qquad (1\ 2\ 3)$$

$$M_{x_1'} = \frac{3\varkappa}{R^5}(C_3 - C)(\Sigma X_i\beta_{i2})(\Sigma X_i\beta_{i3}) \qquad (1.4)$$
$$M_{x_2'} = -\frac{3\varkappa}{R^5}(C_3 - C)(\Sigma X_i\beta_{i1})(\Sigma X_i\beta_{i3}), \qquad M_{x_3'} = 0$$

The mode being studied is determined by a particular solution of the equations of motion (1.2)

$$X_1 = R_0 \cos \omega_1 t, \qquad X_2 = R_0 \sin \omega_1 t, \qquad X_3 = 0 \tag{1.5}$$

$$\varphi_1^{\cdot} = \omega_1 + \omega = \omega^*, \quad q_i = 0, \quad \beta_{ik} = \begin{cases} 1, & i = k \\ 0, & i \neq k \end{cases}$$

$$w_1 = w_2 = 0, \;\; h_1^\circ = h_2^\circ = 0, \;\; h_3^\circ = MR_0^2 \omega_1 + h^\circ \;\; (h^\circ = C_3 \omega^*)$$

Let us consider the problem of stabilizing the motion (1.5) in a limited formulation, i.e. without taking account of perturbations in the coordinates of the gyrostat center of mass X_1, X_2, X_3. Assuming the motion (1.5) to be unperturbed, let the perturbed motion be denoted by

$$q_1, q_2, \omega^* + q_3; \;\; 1 + \beta_{ik} \quad (i = k), \;\; \beta_{ik} \quad (i \neq k)$$
$$w_1, w_2; \; h_1, h_2, h^\circ + h_3$$

(here q_3 denotes the total perturbation in the absolute angular velocity $p_3 = q_3 + \varphi_1^{\cdot}$). Then on the basis of (1.2), (1.4) we obtain the following perturbed motion equations corresponding to (1.5):

$$\begin{aligned} q_1^{\cdot} &= h_{12}q_3 - (h_{13} + \omega^*)\, q_2 + \omega^* \Sigma\, h_{1i}\beta_{i2} + \\ &\quad \beta_{13}\nu \sin 2\omega_1 t + 2\beta_{23}\nu \sin^2 \omega_1 t + \Sigma h_{1i} B_{i1} + v_1 + \\ &\quad 2\nu\, (\beta_{12} \cos \omega_1 t + \beta_{22} \sin \omega_1 t)\, (\beta_{13} \cos \omega_1 t + \beta_{23} \sin \omega_1 t) \end{aligned} \tag{1.6}$$

$$\begin{aligned} q_2^{\cdot} &= (h_{13} + \omega^*)\, q_1 - h_{11}q_3 - \omega^* \Sigma h_{1i}\beta_{i1} - \\ &\quad 2\beta_{13}\nu \cos^2 \omega_1 t - \beta_{23}\, \nu \sin 2\omega_1 t + v_2 + \Sigma h_{1i} B_{i2} - \\ &\quad 2\nu\, (\beta_{11} \cos \omega_1 t + \beta_{21} \sin \omega_1 t)\, (\beta_{13} \cos \omega_1 t + \beta_{23} \sin \omega_1 t) \end{aligned}$$

$$q_3^{\cdot} = h_{31}q_2 - h_{32}q_1 + \Sigma h_{3i} B_{i3}$$

$$\beta_{ii}^{\cdot} = B_{ii} \;\; (i = 1, 2, 3), \quad \beta_{12}^{\cdot} = -q_3 + B_{12}, \quad \beta_{13}^{\cdot} = q_2 + B_{13} \quad (1\;2\;3) \tag{1.7}$$

Here

$$\frac{h_j}{C - J} = h_{1j}, \quad \frac{h_j}{C_3} = h_{3j} \quad (j = 1, 2), \quad \frac{h^\circ + h_3}{C - J} = h_{13}, \quad \frac{h^\circ + h_3}{C_3} = h_{33} \tag{1.8}$$

$$(C - J)\, v_1 = -w_1 + \omega^* h_2, \qquad (C - J)\, v_2 = -w_2 - \omega^* h_1$$

$$B_{i1} = q_3\beta_{i2} - q_2\beta_{i3} \quad (i = 1, 2, 3) \quad (1\,2\,3), \qquad \nu = \frac{3\varkappa}{2R_0^3}\,\frac{C_3 - C}{C - J}$$

It has been established earlier [2] that the controls v_1, v_2, v_3 in the presence of three flyweels can be selected in such a way as to assure asymptotic stability of motion (1.5) in all the phase coordinates of the main body q_i, β_{ik} $(i, k = 1, 2, 3)$ and the minimum of some functional of integral type. It is shown below that an analogous problem in part of the coordinates q_i, β_{ik} has a solution in the presence of just two controls v_1, v_2.

2. Solution of the stabilization problem. Let us seek the controls v_1, v_2 which solve the formulated problem as the sum of two components

$$v_j = v_j^* + v_j^{**} \quad (j = 1, 2) \tag{2.1}$$

where the additional controls v_j^{**} are defined in advance by setting

$$v_1^{**} = -h_{12}q_3 - (\omega^* + q_3)\,(h_{11}\beta_{12} + h_{12}\beta_{22}) \tag{2.2}$$

$$v_2^{**} = h_{11}q_3 + (\omega^* + q_3)(h_{11}\beta_{11} + h_{12}\beta_{21})$$

The controls (2.2) contain only second and third order infinitesimals in β_{jk}, h_j $(j, k = 1, 2)$, q_3. By virtue of (2.1), (2.2) the perturbed motion equations (1.6) become:

$$\begin{aligned} q_1^{\cdot} &= -(\delta + 1)\,\omega^* q_2 + \delta\omega^{*2}\beta_{32} + \beta_{13}\nu \sin 2\omega_1 t + \\ &\quad 2\beta_{23}\nu \sin^2\omega_1 t + v_1^* + Q_1 \\ q_2^{\cdot} &= (\delta + 1)\,\omega^* q_1 - \delta\omega^{*2}\beta_{31} - 2\beta_{13}\nu \cos^2\omega_1 t - \\ &\quad \beta_{23}\nu \sin 2\omega_1 t + v_2^* + Q_2 \\ q_3^{\cdot} &= Q_3 \qquad (\delta = C_3/(C - J)) \end{aligned} \tag{2.3}$$

Here Q_i are second and third order terms in q_i, β_{ik}, $h_i (i, k = 1, 2, 3)$ which are not written down. The functions Q_1, Q_2, Q_3 vanish for $q_1 = q_2 = 0$, $\beta_{13} = \beta_{23} = \beta_{31} = \beta_{32} = 0$.

We pose the following problem: determine the controls v_1^*, v_2^* so that the zero solution of (2.3), (1.7)

$$q_i = 0, \quad \beta_{ik} = 0 \qquad (i, k = 1, 2, 3) \tag{2.4}$$

would be asymptotically stable in the variables q_j, β_{j3}, β_{3j} $(j = 1, 2)$ and the condition of minimum of the functional

$$I = \int_0^\infty \Omega\,(q_1, q_2, q_3; \beta_{11}, \beta_{12}, \ldots, \beta_{33}, v_1^*, v_2^*, t)\, dt \tag{2.5}$$

would hence be satisfied. Here Ω is a positive-definite function in q_j, β_{j3}, β_{3j} $(j = 1, 2)$ which will be found during the solution of the problem on the basis of the Krasovskii and Rumiantsev theorems on optimal stabilization of controlled motions [1, 4].

We construct the optimal control and the function Ω in two steps [2]. First we consider the "shortened" system of perturbed motion equations, and we then generalize the results obtained to the case of the full equations (2.3), (1.7). The shortened system of equations in the stabilized variables q_j, β_{j3}, β_{3j} $(j = 1, 2)$ is

$$\begin{aligned} q_1^{\cdot} &= -(\delta + 1)\,\omega^* q_2 + \delta\omega^{*2}\beta_{32} + \beta_{13}\nu \sin 2\omega_1 t + \\ &\quad 2\beta_{23}\nu \sin^2\omega_1 t + v_1^* \\ q_2^{\cdot} &= (\delta + 1)\,\omega^* q_1 - \delta\omega^{*2}\beta_{31} - 2\beta_{13}\nu \cos^2\omega_1 t - \\ &\quad \beta_{23}\nu \sin 2\omega_1 t + v_2^* \\ \beta_{13}^{\cdot} &= q_2, \quad \beta_{23}^{\cdot} = -q_1, \quad \beta_{31}^{\cdot} = -q_2, \quad \beta_{32}^{\cdot} = q_1 \\ \beta_{33}^{\cdot} &= q_2\beta_{31} - q_1\beta_{32} \end{aligned} \tag{2.6}$$

As Liapunov function we assume

$$\begin{aligned} 2V &= k_1 \sum_{i=1}^{3} \beta_{i3}^2 + k_2 \sum_{i=1}^{3} \beta_{3i}^2 + \sum_{j=1}^{2} m_j q_j^2 + \\ &\quad 2q_1 \sum_{j=1}^{2} (a_{j3}\beta_{j3} + a_{3j}\beta_{3j}) + 2q_2 \sum_{j=1}^{2} (b_{j3}\beta_{j3} + b_{3j}\beta_{3j}) \end{aligned} \tag{2.7}$$

The integrand of the functional (2.5) being minimized we take in the form

$$\Omega_1 = \sum_{j,\,k=1}^{2} e_{jk} q_j q_k + \sum_{j=1}^{2} n_j v_j^{*2} + F(\beta_{13}, \beta_{23}, \beta_{31}, \beta_{32}, t) \tag{2.8}$$

The k_j, m_j, n_j $(j = 1, 2)$ in the functions (2.7), (2.8) are initial positive parameters in terms of which all the remaining coefficients (including the coefficients of the unknown quadratic form F) are expressed; some of the coefficients a_{ik}, b_{ik}, e_{ik} can be periodic functions.

According to the Krasovskii theorem on optimal stabilization [1]

$$v_j^* = -\frac{1}{2n_j}\frac{\partial V}{\partial q_j} \qquad (j = 1, 2) \tag{2.9}$$

we obtain a partial differential equation for the function V

$$\begin{aligned}
&\frac{\partial V}{\partial t} - \frac{1}{4n_1}\left(\frac{\partial V}{\partial q_1}\right)^2 - \frac{1}{4n_2}\left(\frac{\partial V}{\partial q_2}\right)^2 + \frac{\partial V}{\partial q_1}[-(\delta+1)\omega^* q_2 + \\
&\quad \delta\omega^{*2}\beta_{32} + \beta_{13}\nu \sin 2\omega_1 t + 2\beta_{23}\nu \sin^2 \omega_1 t] + \\
&\quad \frac{\partial V}{\partial q_2}[(\delta+1)\omega^* q_1 - \delta\omega^{*2}\beta_{31} - 2\beta_{13}\nu\cos^2\omega_1 t - \beta_{23}\nu \sin 2\omega_1 t] + \\
&\quad \left(\frac{\partial V}{\partial \beta_{13}} - \frac{\partial V}{\partial \beta_{31}}\right) q_2 + \left(\frac{\partial V}{\partial \beta_{32}} - \frac{\partial V}{\partial \beta_{23}}\right) q_1 + \\
&\quad \frac{\partial V}{\partial \beta_{33}} B_{33} + \sum_{j,\,k=1}^{2} e_{jk} q_j q_k + F(\beta_{13}, \beta_{23}, \beta_{31}, \beta_{32}, t) = 0
\end{aligned} \tag{2.10}$$

Substituting (2.9) into (2.10) and extracting coefficients of identical second order terms in $q_k\beta_{j3}, q_k\beta_{3j}$ $(j, k = 1, 2)$, we obtain a system of linear differential equations in $a_{j3}, b_{j3}, a_{3j}, b_{3j}$ $(j = 1,2)$. Assuming $k_j = k, \ m_j = m, \ n_j = n$ $(j = 1, 2)$ for simplicity, we obtain the particular solutions

$$\begin{aligned}
&a_{31} = b_{32} = \mu(\delta+1)\,\omega^*(2k + m\delta\omega^{*2}) \qquad (d = m/n) \\
&a_{32} = -b_{31} = \mu d(2k + m\delta\omega^{*2}) \qquad (\mu = [d^2 + (\delta+1)^2\omega^{*2}]^{-1})
\end{aligned} \tag{2.11}$$

$$\begin{aligned}
&a_{j3} = a_{j3}^* + K_{j3}\cos 2\omega_1 t + L_{j3}\sin 2\omega_1 t \qquad (j = 1, 2) \\
&b_{j3} = b_{j3}^* + M_{j3}\cos 2\omega_1 t + N_{j3}\sin 2\omega_1 t \\
&a_{13}^* = b_{23}^* = -\mu(\delta+1)\,\omega^* m\nu, \quad a_{23}^* = -b_{13}^* = \mu d m\nu \\
&K_{13} = N_{13} = L_{23} = -M_{23} = \mu_1 m\nu\ \{2\omega_1[d^2 + 4\omega_1^2 - (\delta + \\
&\quad 1)\,\omega^{*2}] - (\delta+1)\,\omega^*[d^2 - 4\omega_1^2 + (\delta+1)^2\omega^{*2}]\} \\
&M_{13} = -L_{13} = K_{23} = N_{23} = -\mu_1 d m\nu\,[d^2 + 4\omega_1^2 + (\delta + \\
&\quad 1)^2\omega^{*2} + 4(\delta+1)\,\omega^*\omega_1]
\end{aligned} \tag{2.12}$$

Here

$$\mu_1^{-1} = \begin{vmatrix} -d & 2\omega_1 & (\delta+1)\omega^* & 0 \\ -2\omega_1 & -d & 0 & (\delta+1)\omega^* \\ -(\delta+1)\omega^* & 0 & -d & 2\omega_1 \\ 0 & -(\delta+1)\omega^* & -2\omega_1 & -d \end{vmatrix} \tag{2.13}$$

Furthermore, we find

$$e_{11} = d^2n + a_{23} - a_{32}, \qquad e_{22} = d^2n - b_{13} + b_{31} \tag{2.14}$$
$$2e_{12} = -a_{13} + b_{23}$$

$$F = \frac{1}{4n}\left\{\left[\sum_{j=1}^{2}(a_{j3}\beta_{j3} + a_{3j}\beta_{3j})\right]^2 + \left[\sum_{j=1}^{2}(b_{j3}\beta_{j3} + b_{3j}\beta_{3j})\right]^2\right\} - \tag{2.15}$$
$$\left[\sum_{j=1}^{2}(a_{j3}\beta_{j3} + a_{3j}\beta_{3j})\right](\delta\omega^{*2}\beta_{32} + \beta_{13}\nu\sin 2\omega_1 t +$$
$$2\beta_{23}\nu\sin^2\omega_1 t) + \left[\sum_{j=1}^{2}(b_{j3}\beta_{j3} + b_{3j}\beta_{3j})\right](\delta\omega^{*2}\beta_{31} +$$
$$2\beta_{13}\nu\cos^2\omega_1 t + \beta_{23}\nu\sin 2\omega_1 t)$$

To establish the sign of the function (2.15), let us pass from the dependent variables β_{ik} to the independent variables, the Krylov angles θ, ψ [5], by taking O_1X_2 and Ox_3 as the main axes

$$\beta_{13} = \psi + \ldots, \quad \beta_{31} = -\psi + \ldots, \quad \beta_{23} = -\theta + \ldots, \tag{2.16}$$
$$\beta_{32} = \theta + \ldots$$

The dots denote higher order terms than the first. Considering d sufficiently large, and introducing the small parameter $\varepsilon = 1/d$, let us limit ourselves to the principal first order terms in the found solutions (2.11)–(2.13), i.e. ,

$$a_{13} = \varepsilon m\nu\sin 2\omega_1 t + \ldots, \quad b_{13} = -\varepsilon m\nu(1 + \cos 2\omega_1 t) + \ldots \tag{2.17}$$
$$a_{32} = \varepsilon(2k + m\delta\omega^{*2}) + \ldots, \quad b_{31} = -\varepsilon(2k + m\delta\omega^{*2}) + \ldots$$
$$a_{23} = \varepsilon m\nu(1 - \cos 2\omega_1 t) + \ldots, \quad b_{23} = -\varepsilon m\nu\sin 2\omega_1 t + \ldots$$

By virtue of (2.16), (2.17), the function (2.15) becomes

$$F(\theta, \psi, t) = \frac{\varepsilon}{2m}(2k + m\delta\omega^{*2})(2k - m\delta\omega^{*2})(\theta^2 + \psi^2) + \tag{2.18}$$
$$2\varepsilon m\nu(\delta\omega^{*2} - \nu)(\psi\cos\omega_1 t - \theta\sin\omega_1 t)^2 + \varepsilon^2F^*(\theta, \psi, t) + \ldots$$

Here $F^*(\theta, \psi, t)$ is a quadratic form in the variables θ, ψ with periodic coefficients. Taking account of the smallness of ν from (1.8), the function (2.18) is positive-definite under the conditions $\delta\omega^{*2} < 2k/m, \nu > 0$ or

$$\delta\omega^{*2} < 2k/m, \qquad C_3 > C \tag{2.19}$$

which agrees with results obtained earlier [2]. It is easy to verify that the function V in (2.7) admits an infinitesimal high limit in the variables q_j, β_{j3}, β_{3j} $(j = 1, 2)$ because of (2.17).

By virtue of (2.7), (2.9) the optimal control is written as

$$-v_1^* = dq_1 + \frac{1}{2n}\sum_{j=1}^{2}(a_{j3}\beta_{j3} + a_{3j}\beta_{3j}), \quad -v_2^* = dq_2 + \frac{1}{2n}\sum_{j=1}^{2}(b_{j3}\beta_{j3} + b_{3j}\beta_{3j}) \tag{2.20}$$

On the basis of (1.3), (1.8), (2.1), (2.2), (2.20), we arrive at the following initial control:

$$u_1 = w_1\cos\,\omega^* t + w_2\sin\,\omega^* t, \quad u_2 = -w_1\sin\,\omega^* t + w_2\cos\,\omega^* t \qquad (2.21)$$

$$w_1 = \omega^* h_2 + (C - J)\left[dq_1 + \frac{1}{2n}\sum_{j=1}^{2}(a_{j3}\beta_{j3} + a_{3j}\beta_{3j})\right] +$$
$$h_2 q_3 + (\omega^* + q_3)(h_1\beta_{12} + h_2\beta_{22})$$
$$w_2 = -\omega^* h_1 + (C - J)\left[dq_2 + \frac{1}{2n}\sum_{j=1}^{2}(b_{j3}\beta_{j3} + b_{3j}\beta_{3j})\right] -$$
$$h_1 q_3 - (\omega^* + q_3)(h_1\beta_{11} + h_2\beta_{21})$$

Thus, the control (2.21), (2.11), (2.12) found assures optimal stabilization (in the sense of the minimum (2.5), (2.8)) of the motion (2.4) in the phase coordinates q_j, β_{j3}, β_{j3} ($j = 1, 2$) because of the approximate system of perturbed motion equations (2.6).

Let us note that due to (2.16), from the stabilizability of the motion (2.4) in β_{j3}, β_{3j} ($j = 1, 2$) results the stabilizability of this motion in all the β_{ik} ($i, \ k = 1, \ 2, \ 3$).

It is easy to establish that the Liapunov function (2.7), hence the control (2.21), solves the problem of optimal stabilization of the motion (2.4) by virtue of the complete perturbed motion equations (2.3), (1.7) if the integrand Ω in (2.5) is taken in the form

$$\Omega = \Omega_1 + \Omega_2 \qquad (2.22)$$

Here Ω_1 is the positive-definite function (2.8) in the variables q_j, β_{j3}, β_{3j} ($j = 1, 2$) and Ω_2 denotes the terms of the third and the fourth order of smallness

$$\Omega_2 = -\sum_{j=1}^{2}\left(\frac{\partial V}{\partial q_j}O_j + \frac{\partial V}{\partial \beta_{j3}}B_{j3} + \frac{\partial V}{\partial \beta_{3j}}B_{3j}\right) \qquad (2.23)$$

The conditions of the Rumiantsev theorem will be satisfied (see [4], Theorem 3.1 in the presence of an infinitely small bound in the stabilized variables of the function V) if the higher terms do not violate the sign-definiteness of the basic quadratic form Ω_1. Passing to independent variables by virtue of (2.16), we have for Ω

$$\Omega = \Omega_1^*(q_1, \ q_2, \ \theta, \ \psi) + \Omega_2^*(q_1, \ q_2, \ \theta, \ \psi) + \qquad (2.24)$$
$$q_3\,[f\,(q_1, \ q_2, \ \theta, \ \psi) + \ldots]$$

Here Ω_1^* is a positive-definite quadratic form obtained from (2.8), (2.18), (2.20), and Ω_2^* denotes the terms higher than the second order of smallness which do not influence the sign of Ω; f is an alternating quadratic form of variable sign in q_1, q_2, θ, ψ with coefficients containing the factor q_3. The function Ω in (2.24) is positive-definite in the variables q_1, q_2, θ, ψ if the mentioned coefficients are arbitrarily small [6]. This latter evidently holds if the motion (2.4) is Liapunov-stable relative to q_3. Thus, the control (2.21), (2.11)–(2.13) assures optimal stabilization (in the sense of a minimum of the functional (2.5), (2.24)) of the rotational gyrostat motion (1.5), (2.4) in the phase coordinates q_1, q_2, θ, ψ since this rotation is stable in the angular gyrostat velocity q_3 around the axis of symmetry Ox_3 in the absence of control (the integral $q_3 = \text{const}$ holds).

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STUDY OF THE DYNAMICS OF A SYNCHRONOUS MOTOR BY ASYMPTOTIC METHODS

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N. A. FUFAEV and R. A. CHESNOKOVA
(Gor'kii)

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We investigate the complete system of differential equations describing the dynamics of a synchronous motor with two windings on the rotor, under the assumption that the moment of inertia of the rotor is sufficiently large. We consider two domains of variation of the variable s defining the rotor slippage. In one of them s have finite values, while in the other domain s are small. In the first case we investigate the solutions of the complete system of equations periodic in θ, and in the second case we study the periodic solutions which embrace the state of equilibrium. The conditions of stability of the solutions obtained are given. The stable periodic solutions correspond in the first case to the synchronous modes of the synchronous motor, and in the second case to the oscillations of the rotor relative to the synchronous rate of rotation.

When the transient processes in a synchronous motor are investigated using the complete system of differential equations obtained by Gorev in [1], the following approaches are usually employed: (1) only the equation of the mechanical motion of the rotor is considered [2 – 7]; (2) only the electrical equations are considered, i. e. the transient processes are considered at a constant angular velocity of rotation of the rotor; (3) the complete system of equations is linearized near the steady state motion and small oscillations of the system are studied; (4) the complete system of equations is integrated numerically [1, 8]. However, the dynamics of a synchronous motor as such, has not been investigated to any great extent.

1. The equations of dynamics and statement of the problem. The equations of dynamics of a synchronous motor working in parallel with a network of infinite power, in the driving mode, assume the following form [1] after introducing